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# UNITS IN METABELIAN GROUP RINGS: NON-SPLITTING EXAMPLES FOR NORMALIZED UNITS

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Let $G$ be a finite group and $V(G)$ the normalized units in the integral group ring $\mathbb{Z}G$. That is an element of $V(G)$ is a unit in $\mathbb{Z}G$ which maps to 1 under the augmentation map $\mathbb{Z}G \to \mathbb{Z}$. The question has been raised by Dennis [4] as to when the natural inclusion $G \to V(G)$ is split. Perhaps the main interest in this question is that the group ring problem is answered very instructively, whenever a splitting exists which can be achieved with a map $V(G) \to G$ whose kernel is torsion free.

We give here the first examples which show that the map $G \to V(G)$ does not always split.

We note the following ring-theoretic consequence: Since there exists a splitting $\varphi : V(G) \to G$ if and only if there exists a ring $\mathfrak{R}$ such that $G$ has a normal complement in the units in $\mathfrak{R}$, our examples show that there are groups which can not be split subgroups of the unit group in *any* ring.

This area of units in group rings has been of considerably activity recently [10], [9], [1], [12] and [3]. The most far reaching result is that of Cliff–Sehgal–Weiss:

**Theorem.** *If $G$ has an abelian normal subgroup $H$ with $G/H$ abelian of odd order, then there is a splitting, and it can be achieved by a map $V(G) \to G$ whose kernel is torsion free.*

So the open metabelian cases are groups $G$ which do not have an abelian normal subgroup $H$ with $G/H$ abelian of odd order. Let $G$ be metabelian with an abelian normal subgroup $H$ such that $G/H$ is abelian of even order. We then write $G/H = A_0 \times A_1$, $A_0$ a 2-group and $A_1$ of odd order. Since a possible failure of splitting seems to come from the 2-part of $G/H$, we shall consider the following two cases. ($C_n$ stands for the cyclic group of order $n$.)

*Case* (*i*): $A_1 = 1$. Then $C_8$ must be a subgroup of $A_0$, in order that $V(A_0)$ has non-trivial units – this is necessary for failure of splitting. We shall show that the Frobenius group

$$G = C_{73} \rtimes C_8$$

does not allow a splitting of the natural inclusion $G \to V(G)$.

*Case* (*ii*): $V(A_0)$ has only trivial units; i.e. the exponent of $A_0$ is at most 4. In this case we consider metacyclic Frobenius groups $C_p \rtimes (A_0 \times A_1)$, where $p$ is a prime and $|A_0| \leq 4$. Then both $V(C_p \rtimes A_0)$ and $V(C_p \rtimes A_1)$ allow a splitting as above. However, we shall show that

$$G = C_{241} \rtimes C_{10}$$

does not allow a splitting of the natural inclusion $G \to V(G)$.

These examples indicate that the theorem of Cliff–Sehgal–Weiss is – as a general statement – best possible. However, we would like to point out a positive result, which indicates that a necessary and sufficient condition for a splitting of units of metabelian groups is not so easily obtained.

**Proposition 1.** *Let $G = C_p \rtimes C_{p-1}$ be an affine Frobenius group. Then there exists a splitting $\varphi : V(G) \to G$ with* Ker $\varphi$ *torsion free.*

This was also observed by Sekiguchi [14].

The above proposition is part of the following more general phenomenon: Let $G$ be metabelian with abelian normal subgroup $H$ and abelian quotient $G/H =: B$. Then the conjugation action of $B$ on $H$ induces a homomorphism

$$\mu : B \to \mathrm{Aut}(H).$$

Assume that $\mu$ is surjective, then there is a splitting $\varphi : V(G) \to G$ with Ker $\varphi$ torsion free.

Let us turn to metacyclic Frobenius groups of the form $G_0 = C_p \rtimes C_m$, $p$ a prime. In case $m = p-1$, $G_0$ obviously satisfies the hypothesis of Proposition 1. Hence there exists a splitting

$$\varphi_0 : V(G_0) \to G_0.$$

Let now $G = C_{73} \rtimes C_8$ or $G = C_{241} \rtimes C_{10}$ and $G_0 = C_{73} \rtimes C_{72}$ or $G_0 = C_{241} \rtimes C_{240}$. Then $G \leq G_0$ and

$$\mathrm{Im}\, \varphi_0 |_{V(G)} \supsetneq G,$$

because, if $\mathrm{Im}\, \varphi_0 |_{V(G)} \subset G$, then $\varphi |_{V(G)}$ would be a splitting; but no such splitting exists.

*This note is organized as follows:* In Section 1 we prove the proposition and consider splittings 'which come from ideals', a concept also used in [3]. Using this we show that for our groups $C_{73} \rtimes C_8$ and $C_{241} \rtimes C_{10}$, a splitting can not come from an ideal. In Section 2 we use the congruence subgroup theorem of Bass–Milnor–Serre [2] to show that every splitting must essentially come from an ideal. Most of our results should hold for metabelian groups in general; however, we did not elaborate on this, since we are interested in particular metacyclic groups.

*Erratum:* The first author wants to point out a mistake in [11] where he claimed that metacyclic Frobenius groups always allow a splitting. The point is that the sequences $\mathfrak{E}_1$ and $\mathfrak{E}_2$ [11, p. 264] are not equivalent. The mistake was also noted by S.K. Sehgal.

We remark that in the proof of the theorem of Cliff–Sehgal–Weiss [3] one of the main difficulties is to prove that $V(G) \to G$ has a torsion-free kernel. This was essentially proved in Jackson's thesis of 1968 [7] and brought up to date in [12]. According to a letter of Sekiguchi [14], Miyata has done the same.

## 1. Splitting by ideals

Throughout $G = H \rtimes C$ is a metacyclic Frobenius group,

$$\begin{aligned} &H = \langle h_0 : h_0^p = 1 \rangle, \quad p \geq 7,\ p \text{ a prime}, \\ &C = \langle c : c^m = 1 \rangle, \quad m > 2,\ m \mid (p-1). \end{aligned} \tag{1}$$

The split exact sequence

$$\mathfrak{E}: \quad 1 \to H \to G \to C \to 1$$

gives rise to the split exact sequence of normalized units

$$\mathfrak{E}_V: \quad 1 \to V_H(G) \to V(G) \to V(C) \to 1, \tag{$*$}$$

where

$$V_H(G) = \{1 + x : x \in \mathfrak{y}\!\uparrow^G\} \cap V(G),$$

and $\mathfrak{y}\!\uparrow^G = \mathfrak{y}\mathbb{Z}H \simeq \mathfrak{y} \otimes_{\mathbb{Z}H} \mathbb{Z}G$, $\mathfrak{y}$ being the augmentation ideal $\mathbb{Z}H$.

We first recall some results on the structure of $\mathbb{Z}G$ – for the details we refer to [5], [6], [13]. (These results can also be proved by computing the index of $\mathbb{Z}G$ in a maximal order, as was done by H. Jacobinski (unpublished).)

Let $e_0 = 1 - |H|^{-1} \sum_{h \in H} h$, then $e_0$ is a central primitive idempotent in $\mathbb{Q}G$ and

$$\mathbb{Z}Ge_0 = \begin{pmatrix} R & \cdots & R \\ \mathfrak{p} & \ddots & \vdots \\ \vdots & \ddots & \\ \mathfrak{p} & \cdots \ \mathfrak{p} & R \end{pmatrix}_m,$$

where $R = \operatorname{Fix}_C \mathbb{Z}[\sqrt[p]{1}]$, by viewing $C$ as a subgroup of $\operatorname{Gal}(\mathbb{Q}(\sqrt[p]{1})/\mathbb{Q})$, and $\mathfrak{p}$ is the unique maximal ideal in $R$ above $p$, with $R/\mathfrak{p}=\mathbb{F}_p$ the field with $p$ elements.

In particular, the natural representation of $G$ in $\mathbb{Z}Ge_0$ has the form

$$\nu(g)=\begin{pmatrix} \nu_1(g) & & \cdots & \\ \vdots & & \nu_2(g) \;\; * & \vdots \\ & ** & \ddots & \\ & \cdots & & \nu_m(g) \end{pmatrix}_m \tag{2}$$

where the elements in $*$ lie in $R$ and those in $**$ lie in $\mathfrak{p}$.

Moreover, easy computations show

$$\mathfrak{y}\!\uparrow^G=\begin{pmatrix} \mathfrak{p} & R & \cdots & R \\ \vdots & & \ddots & \vdots \\ & & & R \\ \mathfrak{p} & & \cdots & \mathfrak{p} \end{pmatrix}_m ; \qquad \mathfrak{y}^2\!\uparrow^G=\begin{pmatrix} \mathfrak{p} & \mathfrak{p} & R & \cdots & R \\ \vdots & & \ddots & \ddots & \vdots \\ & & & \ddots & R \\ \mathfrak{p} & & & & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \cdots & & \mathfrak{p} \end{pmatrix}_m . \tag{3}$$

and so

$$\mathfrak{y}\!\uparrow^G/\mathfrak{y}^2\!\uparrow^G=\begin{pmatrix} \mathbb{F}_p & & \\ & \ddots & \\ & & \mathbb{F}_p \\ \bar{\mathfrak{p}} & & \end{pmatrix}_m , \quad \bar{\mathfrak{p}}=\mathfrak{p}/\mathfrak{p}^2.$$

Let $I$ be a two-sided $\mathbb{Z}G$-ideal contained in $\mathfrak{y}\!\uparrow^G$. Then we put

$$V_I(G)=\{1+y : y\in I\}\cap V(G), \tag{4a}$$

so that

$$V_H(G)=V_{\mathfrak{y}\uparrow^G}(G). \tag{4b}$$

**Claim 1.** (i) *We have a natural isomorphism*

$$\sigma : \mathfrak{y}\!\uparrow^G/\mathfrak{y}^2\!\uparrow^G \simeq V_H(G)/V_{\mathfrak{y}^2\uparrow^G}(G),$$

*which is equivariant with respect to the conjugation action of* $V(G)$.

(ii) *The composition factors of* $\mathfrak{y}\!\uparrow^G/\mathfrak{y}^2\!\uparrow^G$ *under C-conjugation are isomorphic to H with C-conjugation.*

(iii) $\nu_i\nu_{i+1}^{-1}|_C$ *modulo* $\mathfrak{p}$ *is the conjugation representation of C on* $H=\mathbb{F}_p$, $1\le i\le n$, $n+1=1$.

**Proof.** (i) We define

$$\sigma^{-1} : V_H(G)/V_{\mathfrak{y}^2\uparrow^G}(G)\to \mathfrak{y}\!\uparrow^G/\mathfrak{y}^2\!\uparrow^G$$

by

$$(1+x)V_{\mathfrak{y}^2\uparrow^G}(G)\mapsto x+\mathfrak{y}^2\!\uparrow^G.$$

Then $\sigma^{-1}$ is well defined by (4), and it is a group homomorphism since

$$(1+x)(1+y)=(1+x+y)+xy \quad \text{and} \quad xy\in\mathfrak{y}^2\!\uparrow^G.$$

Moreover, it is injective by the construction of $V_{\mathfrak{y}^2\uparrow^G}(G)$, and $\sigma^{-1}$ surely is equivariant with respect to conjugation by $V(G)$.

By [5, 3.5], $V_I(G)=\{1+x, x\in I, \det 1+x\in R^*\}$, where $R^*$ denotes the units in $R$. Hence

$$|V_H(G)/V_{\mathfrak{y}^2\uparrow^G}(G)|=p^m=|\mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G|.$$

Thus $\sigma^{-1}$ is an isomorphism.

(ii) As $C$-module under conjugation we have

$$\mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G=\bigoplus_{i=0}^{m}\{(h-1)c^i\}_{h\in H},$$

and so all composition factors of $\mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G$ are isomorphic to $H$ as $C$-modules via conjugation.

(iii) We now compute the action using the representation $v$. If an element of $\mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G$ is represented by

$$x=\begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ \vdots & \ddots & a_2 & & \vdots \\ & & & \ddots & 0 \\ 0 & & & \ddots & a_{m-1} \\ a_m & 0 & \cdots & & 0 \end{pmatrix},$$

then an easy computation shows

$$v(c)xv(x^{-1})=$$

$$\begin{pmatrix} 0 & \ddots & v_1(c)v_2(c^{-1})a_1 & 0 & \cdots & 0 \\ 0 & & & & \ddots & \\ \vdots & & \ddots & & & \vdots \\ & & & & \ddots & \\ 0 & & & \ddots & & 0 \\ v_m(c)v_1(c^{-1})a_m & & 0 & \cdots & 0 & v_{m-1}(c)v_m(c^{-1})a_{m-1} \end{pmatrix}.$$

Since $v$ is a representation, we have $v_i(c^{-1})=v_i(c)^{-1}$, and by (ii) all $C$-composition factors of $\mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G$ are isomorphic to $H$ as $C$-module, so the statement follows. □

We now turn to *splittings which come from ideals*. All the previous splittings – including [3] – were constructed in the following way: Look at the two-sided $\mathbb{Z}G$-ideals $I$ with $\mathfrak{y}^2\uparrow^G\subset I\subset\mathfrak{y}\uparrow^G$ and $\mathfrak{y}\uparrow^G/I\simeq H$ as abelian groups. Then by Claim 1, $V_I(G)$ is a normal subgroup of $V(G)$, and $V_H(G)/V_I(G)\simeq H$ as abelian groups. The sequence $\mathfrak{E}_V(*)$ gives rise to the exact sequence

$$\bar{\mathfrak{E}}_V:\quad 1\to V_H(G)/V_I(G)\to V(G)/V_I(G)\to V(C)\to 1.$$

Since $V_H(G)/V_I(G)\underset{\text{nat}}{\simeq}\mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G$ is abelian, $V(C)$ acts on $V_H(G)/V_I(G)$ by conjugation. This conjugation is induced from the conjugation of $V(G)$ on $V_H(G)$; hence the canonical map

$$\varphi' : V_H(G) \to V_H(G)/V_I(G)$$

is $V(G)$-equivariant. The action of $V(C)$ on $V_H(G)/V_I(G) \simeq H$ leads to a group homomorphism

$$\mu_I : V(C) \to \mathrm{Aut}(H). \tag{5}$$

On the other hand, $C$ acts on $H$ by conjugation, and so we have a homomorphism

$$\mu : C \to \mathrm{Aut}(H).$$

**Claim 2.** *Let $I$ be as above, and assume* $\mathrm{Im}\,\mu_I \subset \mathrm{Im}\,\mu$, *then there is a splitting for the natural injection $G \to V(G)$.*

**Proof.** By Higman's theorem [6],

$$V(C) \simeq C \times T,$$

where $T$ is free abelian of finite rank. Since $\mathrm{Im}\,\mu_I \subset \mathrm{Im}\,\mu \simeq C$ we can choose the complement $T$ in such a way that

$$T \subset \mathrm{Ker}\,\mu_I.$$

Let now

$$\bar{\varphi} : V(C) \to C$$

be the projection with $\mathrm{Ker}\,\bar{\varphi} = T$. We then form the pullback along $\bar{\varphi}$:

$$\begin{array}{lccccccccc}
\mathfrak{E}: & 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & C & \longrightarrow & 1 \\
 & & & \| & & \uparrow & & \uparrow{\scriptstyle \bar{\varphi}} & & \\
\mathfrak{E}_{\bar{\varphi}}: & 1 & \longrightarrow & H & \longrightarrow & X & \longrightarrow & V(C) & \longrightarrow & 1
\end{array}$$

Then $\mathfrak{E}_{\bar{\varphi}}$ is split exact, and $V(C)$ acts via conjugation on $H$ in the same way as does $\bar{\varphi}(V(C))$. On the other hand we have the pushout along $\varphi'$:

$$\begin{array}{lccccccccc}
\mathfrak{E}_V: & 1 & \longrightarrow & V_H(G) & \longrightarrow & V(G) & \longrightarrow & V(C) & \longrightarrow & 1 \\
 & & & \downarrow{\scriptstyle \varphi'} & & \downarrow & & \| & & \\
{}_{\varphi'}\mathfrak{E}_V: & 1 & \longrightarrow & H & \longrightarrow & Y & \longrightarrow & V(C) & \longrightarrow & 1
\end{array}$$

In ${}_{\varphi'}\mathfrak{E}_V$ the group $V(C)$ acts on $H$ via $\mu_I$; but since by construction $T \subset \mathrm{Ker}\,\mu_I$, and since $\mu_I|_C = \mu$, $V(C)$ acts in the same way in both sequences ${}_{\varphi'}\mathfrak{E}_V$ and $\mathfrak{E}_{\bar{\varphi}}$. But both sequences are split and so they are equivalent; i.e. we get a combined morphism

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & C & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \varphi'} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \bar{\varphi}} & & \\
1 & \longrightarrow & V_H(G) & \longrightarrow & V(G) & \longrightarrow & V(C) & \longrightarrow & 1
\end{array} \qquad \text{(D)}$$

and hence we have constructed a splitting $\varphi$ of the natural injection $G \to V(G)$. (We note that by [7], [12] Ker $\varphi$ is torsion free.) □

We say that a *splitting* $\varphi: V(G) \to G$ *comes from an ideal* $I$, if $\varphi$ renders the diagram (D) commutative and $\varphi'$ is induced from a $\mathbb{Z}G$-homomorphism $\mathfrak{y}\uparrow^G/I \simeq H$ as above. In this case one must necessarily have Im $\mu_I \subset$ Im $_\mu$. Hence for a splitting $\varphi$ to come from an ideal $I$ is necessary and sufficient that Im $\mu_I \subset$ Im $\mu$.

The *Proposition* 1 follows now immediately from Claim 2.

In order to get a workable condition, we return to the representation $\nu$ of (2).

Because of the isomorphism

$$\mathbb{Z}G\varepsilon_0/\mathfrak{y}\uparrow^G \simeq \mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G \simeq \mathbb{F}_pC,$$

and because of Claim 1(iii), $\nu_i|_C$, $1 \le i \le m$, modulo $\mathfrak{p}$ are the different irreducible representations in $\mathbb{F}_pC$. By Claim1(iii), $\nu_i(c)/\nu_{i+1}^{-1}(c)$ modulo $\mathfrak{p}$ is independent of $i$, say it is multiplication by $\alpha$, a primitive $m$-th root of unity in $\mathbb{F}_p$. Thus, modulo $\mathfrak{p}$ we have

$$\nu_i(c) = \nu_{i+1}(c)\alpha.$$

Since $\nu_i$ modulo $\mathfrak{p}$ are the different irreducible representations, there exists an $i_0$ with

$$\nu_{i_0}(c) = \alpha \quad \text{modulo } \mathfrak{p}.$$

Hence after renumbering by passing to a sublattice of $M_0 = (R, \dots, R)^t$, on which $G$ acts via $\mathbb{Z}Ge_0$, we may assume

$$\begin{aligned}&\nu_i \text{ modulo } \mathfrak{p} \text{ is induced from the representation}\\ &c \mapsto \alpha^i, \quad 1 \le i \le n.\end{aligned} \tag{6}$$

If we denote by $I_i$ the kernels of the map $\mathfrak{n}\uparrow^G \to \mathbb{F}_p^i$, $1 \le i \le m$, then these are two sided ideals, and so we can apply Claim 2.

**Claim 3.** *There exists a splitting of* $G \to V(G)$ *coming from an ideal* $I$ *if and only if there exists an* $i$, $1 \le i \le m$, *such that*

$$\nu_i(u)\nu_{i+1}^{-1}(u)$$

*modulo* $\mathfrak{p}$ *has order dividing* $m$ *for every* $u \in V(C)$.

**Proof of Claim 3.** Let $I$ be a two-sided ideal in $\mathfrak{y}\uparrow^G$ with $\mathfrak{y}\uparrow^G/I \simeq H$, then by Claim 2, a splitting comes from an ideal $I$ if and only if Im $\mu_I \subset$ Im $\mu$; i.e. the elements in

$V(C)$ have to act on $H$ with order dividing $m$, where the action is conjugation. Now, above we have found a decomposition of $\mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G$ as a direct sum of modules, on which $u \in V(C)$ acts as $v_i(u)v_{i+1}^{-1}(u)$, and hence by the Jordan–Hölder theorem, the action of $u$ on $\mathfrak{y}\uparrow^G/I$ must be the same as the action of $v_i(u)/v_{i+1}^{-1}(u)$ on $\mathbb{F}_p$ for some $1 \le i \le m$. $\square$

As an immediate consequence we obtain

**Lemma 1.** *If $m=p-1$, then each of the ideals $I_i$, $1 \le i \le p-1$, induces a splitting.*

**Lemma 2.** *If $m$ is odd, then $I_{i_0}$ gives a splitting for $i_0 = \frac{1}{2}(m-1)$.*

**Proof.** We have a decomposition $V(C) = T \times C$, where $T$ consists of real units, $m$ being odd [8, Ch. III, §4, 4.1]. Hence if $*: \mathbb{Z}C \to \mathbb{Z}C$ is the involution induced by $c \mapsto c^{-1}$, then the elements in $T$ are invariant under $*$. (This is proved in more generality in [3].) If now $u = \sum z_i c_i \in V(C)$, then by a proper choice of $v_1$,

$$v_{(m-1)/2}(u) = v_1\left(\sum z_i c_i^{(m-1)/2}\right)$$

and

$$v_{(m-1)/2+1}(u) = v_1\left(\sum z_i c_i^{(m-1)/2+1}\right),$$

i.e.

$$v_{(m-1)/2}(u) = v_{(m-1)/2+1}(u^*),$$

but for $u \in T$ we have $u = u^*$, and so

$$v_{(m-1)/2}(u)v_{(m-1)/2+1}^{-1}(u) = 1,$$

i.e. $I_{(m-1)/2}$ gives a splitting by Claim 3.

**Remark.** In general this is not the only splitting, as the following example shows: Let $G = C_{11} \rtimes C_5$, then $u = c + c^{-1} - 1$ is a unit in $\mathbb{Z}C_5$ which generates a complement in $V(C_5)$ to $C_5$. If we put $\alpha = 4$ in $\mathbb{F}_{11}$ and $v_i : c \to \alpha^i$, then we have the following congruences modulo 11:

$$v_1(u) \equiv 6, \quad v_2(u) \equiv 2, \quad v_3(u) \equiv 2,$$

$$v_4(u) \equiv 6, \quad v_5(u) \equiv 1.$$

Hence for the quotients we have

$$v_1(u)/v_2^{-1}(u) \equiv 3, \quad v_2(u)/v_3^{-1}(u) \equiv 1, \quad v_3(u)/v_4^{-1}(u) \equiv 4,$$

$$v_4(u)/v_5^{-1}(u) \equiv 6, \quad v_5(u)/v_1^{-1}(u) \equiv 2.$$

Hence apart from the ideal $I_2$ (Lemma 2), also the ideals $I_1$ and $I_3$ yield splittings; but $I_4$ and $I_5$ can not be used to obtain splittings.

We now turn to our examples.

**Example 1.** Let $G = C_{73} \rtimes C_8$ and $C_8 = \langle c : c^8 = 1 \rangle$.

**Claim 4.** *The element $u = 2 + c - c^3 - c^4 - c^5 + c^7$ is a unit in $\mathbb{Z}C_8$.*

We first need:

**Lemma 3.** *Let $R$ be an integral domain with field of quotients $K$ and $\Lambda \subset \Gamma$ $R$-orders in the finite dimensional $K$-algebra $A$. If $u \in \Lambda$ is a unit in $\Gamma$, then $u$ is a unit in $\Lambda$.*

**Proof of Lemma 3.** Let $\iota : \Lambda \to \Gamma$ be the inclusion, and denote by $\nu_u$ the multiplication by $u$. Then we have the diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \Lambda & \xrightarrow{\nu_u} & \Lambda & \longrightarrow & \bar{\Lambda} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \| & & \\ 0 & \longrightarrow & \Gamma & \xrightarrow{\nu_u} & \Gamma & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Since $\operatorname{Coker} \iota$ is artinian, $\nu_u$ induces an automorphism on it and so $\bar{\Lambda} = 0$ by the 'snake' lemma. □

**Proof of Claim 4.** Let

$$\Gamma = \mathbb{Z} \prod \mathbb{Z} \prod \mathbb{Z}[i] \prod \mathbb{Z}[\zeta]$$

be the maximal order containing $\mathbb{Z}C_8$, where $\zeta$ is a primitive 8-th root of unity. Then the generator $c$ of $C_8$ is represented as

$$c \mapsto (1, -1, \mathrm{i}, \zeta),$$

and because of the above lemma, it suffices to show that $u$ is a unit in $\Gamma$. We note that $1 + \zeta + \zeta^{-1}$ is a unit with inverse $(-1 + \zeta + \zeta^{-1})$ and

$$(1 + \zeta + \zeta^{-1})^2 = 2 + \zeta - \zeta^3 - \zeta^4 - \zeta^5 + \zeta^7.$$

Now since $u$ is represented as

$$u \mapsto (1, 1, 1, 2 + \zeta - \zeta^3 - \zeta^4 - \zeta^5 + \zeta^7),$$

it is a unit in $\mathbb{Z}C_8$. □

We now apply the above procedure: Since 10 is a primitive 8-th root of unity on $\mathbb{F}_{73}$, we have

$$\nu_1 : c \to 10,$$

for suitable choice of $c$ and so an easy computation shows that

$$\begin{aligned} v_1(u) &\equiv -6 \mod(73), & v_2(u) &\equiv 1 \mod(73), \\ v_3(u) &\equiv 12 \mod(73), & v_4(u) &\equiv 1 \mod(73), \\ v_5(u) &\equiv 12 \mod(73), & v_6(u) &\equiv 1 \mod(73), \\ v_7(u) &\equiv -6 \mod(73), & v_8(u) &\equiv 1 \mod(73). \end{aligned}$$

Hence we obtain for the elements $v_i(u)v_{i+1}(u)^{-1}$ either 12 or $-6 \equiv 12^{-1}$ modulo 73. But both elements have order 36 modulo 73. Hence these calculations show, that for the group $G = C_{73} \rtimes C_8$ no ideal can yield a splitting.

**Example 2.** Let $G = C_{241} \rtimes C_{10}$ with $C_{10} = \langle c : c^{10} = 1 \rangle$. Then by the above remarks, $G_1 = C_{241} \rtimes C_2$ and $G_2 = C_{241} \rtimes C_5$ allow splittings which come from ideals, $\mathbb{Z}G_2$ has no non-trivial units. Hence it is necessary to construct a unit $u \in V(C_{10})$ which maps to the identity under $V(C_{10}) \to V(C_5) \times V(C_2)$.

**Claim 5.** *The element*

$$u = -372099\, c^0 + 114985\, c^1 + 301035\, c^2 - 301035\, c^3 - 114985\, c^4$$
$$+372100\, c^5 - 114985\, c^6 - 301035\, c^7 + 301035\, c^8 + 114985\, c^9$$

*is a unit in* $\mathbb{Z}C_{10}$ *which maps to the identity under* $V(C_{10}) \to V(C_5) \times V(C_2)$.

**Proof.** This can be proved by 'brute force'; however, that does not show how we found $u$. Hence we take some space explaining it. The maximal order $\Gamma$ containing $\mathbb{Z}C_{10}$ is

$$\Gamma = \mathbb{Z} \prod \mathbb{Z} \prod \mathbb{Z}[\zeta] \prod \mathbb{Z}[\zeta],$$

where $\zeta$ is a primitive 5-th root of unity. We choose the following embedding

$$\mathbb{Z}C_{10} \to \mathbb{Z} \prod \mathbb{Z} \prod \mathbb{Z}[\zeta] \prod \mathbb{Z}\zeta], \qquad c \mapsto (1, -1, \zeta, -\zeta).$$

If $\alpha : \mathbb{Z}C_{10} \to \mathbb{Z}C_2$ and $\beta : \mathbb{Z}C_{10} \to \mathbb{Z}C_5$ are the natural projections, then

$$\operatorname{Ker} \alpha \cap \operatorname{Ker} \beta \supset (0, 0, 0, 2(1-\zeta)\mathbb{Z}[\zeta]),$$

and we have

$$(c^6 - 1)(c^5 - 1) = 1 + c - c^5 - c^6 = (0, 0, 0, 2(1-\zeta)).$$

Now $u_0 = (\zeta^2 + \zeta^3 - 1)$ is a unit in $\mathbb{Z}[\zeta]$ with inverse $u_0^{-1} = (\zeta + \zeta^4 - 1)$. We are looking for a power $u$ such that $u_0^n - 1 \in 2(1-\zeta)\mathbb{Z}[\zeta]$. $n = 15$ will do, and

$$u_0^{15} = -514229 + 832040\, \zeta^2 + 832040\, \zeta^3$$

is congruent to 1 modulo $2(1-\zeta)\mathbb{Z}[\zeta]$.

Observing

$$5=(1-\zeta)^4\zeta^3(1-\zeta^2-\zeta^3),$$

we can write $u_0^{15}$ in the above form as an element in $\mathbb{Z}C_{10}$. □

In order to apply Claim 3, there is no loss of generality if we assume (cf. (6))

$$\nu_1 : c \mapsto 36 \quad \text{in } \mathbb{F}_{241}, \text{ modulo } \mathfrak{p}.$$

We put $\upsilon = c - c^5 + c^9$, then $\upsilon \in V(C_{10})$ with inverse $c^3 - c^5 + c^7$.

Assume now that for $G = C_{241} \rtimes C_{10}$ there would exist a splitting induced by an ideal. Then according to Claim 3, there exists an $i$, $1 \le i \le 10$, such that modulo $\mathfrak{p}$ both

$$\nu_i(u)\nu_{i+1}^{-1}(u) \quad \text{and} \quad \nu_i(\upsilon)\nu_{i+1}^{-1}(\upsilon) \tag{7}$$

must have order in $\mathbb{F}_{241}$ dividing 10. We put

$$\kappa_i = \nu_i(u)\nu_{i+1}^{-1}(u) \quad \text{and} \quad \lambda_i = \nu_i(\upsilon)\nu_{i+1}^{-1}(\upsilon).$$

Then by computation we obtain the list shown in Table 1.

Table 1

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\kappa_i$ | 233 | 233 | 30 | 1 | 1 | 233 | 30 | 30 | 233 | 30 |
| $\lambda_i$ | 151 | −1 | 83 | 191 | 53 | 151 | −1 | 83 | 191 | 53 |

Hence $u$ satisfies (7) only for $i = 4, 5$ and $\upsilon$ satisfies (7) only for $i = 2, 7$. It should be noted that the above elements have the following order in $\mathbb{F}_{241}$:

$$|233| = |30| = 8 \quad \text{and} \quad 233^{-1} = 30,$$

$$|151| = |83| = 60 \quad \text{and} \quad 151^{-1} = 83,$$

$$|191| = |53| = 120 \quad \text{and} \quad 191^{-1} = 53.$$

Hence the table shows that for $G = C_{241} \rtimes C_{10}$ there can not exist a splitting induced by an ideal.

## 2. Congruence subgroup results

Again in this section $G = H \rtimes C$ is a Frobenius group $H = \langle h_0 : h_0^p = 1 \rangle$, for a prime $p \ge 7$ and $C = \langle c_0 : c_0^m = 1 \rangle$ with $2 < m \mid (p-1)$. The aim of this section is to show that if

$$\varphi : V(G) \to G$$

splits the natural inclusion, then it must essentially come from an ideal; i.e. the splittings described in the previous section are the only ones possible.

**Proposition 2.** *If $\varphi : V(G) \to G$ is a splitting as above, then $\varphi$ renders the following diagram commutative:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & C & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \varphi'} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi''} & & \\
1 & \longrightarrow & V_H(G) & \longrightarrow & V(G) & \longrightarrow & V(C) & \longrightarrow & 1
\end{array}$$

*where $\varphi' = \varphi|_{V_H(G)}$ and $V_{\mathfrak{n}^2\uparrow G}(G) \subset \operatorname{Ker} \varphi'$.*

The main tool in the proof is the use of Bass–Milnor–Serre's congruence subgroup theorem [2], and we are indebted to H. Bass for pointing out that the kernel of $\varphi'$ must contain a congruence subgroup.

The *proof* of the proposition will be done in several steps.

**Claim 6.** $\operatorname{Im} \varphi|_{V_H(G)} = H$.

**Proof.** By [12], we can write

$$V_H(G) = T_1 \rtimes H \tag{8}$$

as a semidirect product with $T_1 = V_{\mathfrak{n}\mathfrak{g}}(G)$ a torsion-free group. We consider $\varphi' = \varphi|_{V_H(G)}$. Since $V(G)/V_H(G)$ is abelian, $\operatorname{Im} \varphi' \supset H$, and so $\operatorname{Im} \varphi' = H \rtimes \tilde{C}$ for a subgroup $\tilde{C}$ of $C$. Moreover, $\varphi'(1 + (h-1)) = h$ since $\varphi$ is a splitting and $H \leq V_H(G)$. Thus $(\varphi')^{-1}(H) \supset H$ and so

$$(\varphi')^{-1}(H) = T_2 \rtimes H.$$

But then

$$T_1/T_2 \rtimes H \simeq H \rtimes \tilde{C};$$

however, such an isomorphism can only exist if $\tilde{C} = 1$, since in one group $H$ is a complement, in the other group, $H$ is a Frobenius kernel. □

We note that by means of Claim 6, the above diagram is commutative, and it remains to show $V_{\mathfrak{n}^2\uparrow G}(G) \subset \operatorname{Ker} \varphi'$.

We now have to introduce some more *notations:* If $\mathfrak{a} \neq 0$ is an ideal in $R$ (cf. (2)), we denote by $\Gamma_m(\mathfrak{a})$ the congruence subgroup in $\mathrm{SL}_m(R)$ with respect to $\mathfrak{a}$; i.e. the kernel of the natural map

$$\mathrm{SL}_m(R) \to \mathrm{SL}_m(R/\mathfrak{a}).$$

We denote by $e_{ij}$ the $m \times m$ matrix with 1 at the $(i,j)$-position and 0 elsewhere. $E_m(\mathfrak{a})$ is the normal subgroup of $\mathrm{SL}_m(R)$ generated by the $\mathfrak{a}$-elementary matrices $E + \alpha e_{ij}$, $\alpha \in \mathfrak{a}$, $i \neq j$, where $E$ denotes the $m \times m$ identity matrix.

For an ideal $I \subset \mathfrak{p}\uparrow^G$ we put

$$\mathrm{SV}_I(G) = \{u \in V_I(G)\colon \det u = 1\} \\ = V_I(G) \cap \mathrm{SL}_m(R) \qquad (\text{cf. (3)}). \tag{9}$$

If $R_1^*$ are the units in $R$ congruent to 1 modulo $\mathfrak{p}$, then

$$R_1^* \simeq \tilde{R}_1^* = \left\{ \begin{pmatrix} 1+x & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_m, x \in \mathfrak{p} \right\} \cap \mathrm{GL}_m(R) \tag{10}$$

is a subgroup of $V_H(G)$ (cf. (3)), and

$$V_H(G) = \mathrm{SV}_H(G)\tilde{R}_1^*.$$

Since $m \neq 1$, the only roots of unity in $R$ are $\pm 1$, and so the elements in $H_1 < V_H(G)$ (cf. (8)) lie in $\mathrm{SV}_H(G)$; i.e.

$$\varphi'|_{\mathrm{SV}_{H(G)}} : \mathrm{SV}_H(G) \to H$$

is surjective. We put

$$\varphi'_S = \varphi'|_{\mathrm{SV}_{H(G)}}. \tag{11}$$

**Claim 7.** Ker $\varphi'_S$ *contains the congruence subgroup* $\Gamma_m(\mathfrak{p}^2)$.

**Proof.** Using the relation

$$[1 + \alpha e_{ik}, 1 + \beta e_{kj}] = 1 + \alpha\beta e_{ij}$$

for $i, j, k$ distinct, note $m > 2$, and we obtain – observing that $H$ is abelian – that the subgroup

$W$ of $\mathrm{SL}_m(R)$ generated by

$$\{1 + re_{ij} : j - i \geq 2, r \in R\} \\ \cup \{1 + \alpha e_{r,s}, r \neq s, -(m-2) \leq s - r \leq 1, \alpha \in \mathfrak{p}\} \\ \cup \{1 + \beta e_{m,1}, \beta \in \mathfrak{p}^2\} \tag{12}$$

lies in the kernel of $\varphi'_S$.

In particular, $E_m(\mathfrak{p}^2) \subset \mathrm{Ker}\, \varphi'_S =: N$. Now, since the only units of finite order in $R$ are $\pm 1$, we conclude with [2, 3.6]

$$|\Gamma_m(\mathfrak{p}^2) : E_m(\mathfrak{p}^2)| \leq 2.$$

On the other hand, $\mathrm{SV}_H(G) \supset \Gamma_m(\mathfrak{p})$ and since $|\mathrm{SV}_H(G) : N|$ is odd, we have

$$\Gamma_m(\mathfrak{p}^2) \subset N,$$

as claimed. □

The remaining part of the *proof of Proposition* 2 concerns only questions about some very common finite groups.

**Claim 8.** $\mathrm{SV}_{\mathfrak{p}^2\uparrow G}(G) \subset \operatorname{Ker} \varphi'_S =: N$.

**Proof.** By Claim 7 and the fact that $\mathrm{SL}_m(R) \to \mathrm{SL}_m(R/\mathfrak{p}^2)$ is surjective (both groups are generated by elementary matrices) we are reduced to a question about subgroups of $\mathrm{SL}_m(R/\mathfrak{p}^2)$.

We view the latter as an extension

$$0 \to M \to \mathrm{SL}_m(R/\mathfrak{p}^2) \to \mathrm{SL}_m(R/\mathfrak{p}) \to 1,$$

where $M = \mathrm{sl}_m(R/\mathfrak{p})$ is the module consisting of $m \times m$ matrices of trace zero. The corresponding sequence of the image of $\mathrm{SV}_H(G)$ in $\mathrm{SL}_m(R/\mathfrak{p}^2)$, denoted by $\overline{\mathrm{SV}_H(G)}$, is

$$0 \to M \to \overline{\mathrm{SV}_H(G)} \to U \to 1,$$

where

$$U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{m \times n}$$

is the unipotent radical of the standard Borel subgroup of $\mathrm{SL}_m(\mathfrak{p})$. Let $M'$ be the submodule of $M$ consisting of the matrices with zero in the lower left hand corner ($(m,1)$-position), and put $M''$ to be the submodule of $M'$ consisting of the matrices of $M'$ with zero diagonal entries. Then the subgroup $W$ defined in (12) has image in $\mathrm{SL}_m(R/\mathfrak{p}^2)$ containing $M''$. Since the conjugate of $e_{i,i+1}$ under $1 + e_{i+1,i}$ is

$$e_{i,i+1} + e_{i+1,i+1} - e_{ii} - e_{i+1,i},$$

the $W$-submodule of $M'$ generated by $M''$ contains the generators $e_{i+1,i+1} - e_{ii}$ of the trace zero matrices, and thus all of $M'$. Since $W$ contains $1 + Re_{ij}$ for $j - i \geq 2$ and because of the structure of $\mathfrak{n}^2\uparrow^G$ (cf. (3)), the claim follows easily. □

**Claim 9.** $\operatorname{Ker} \varphi' = \operatorname{Ker} \varphi \,|_{V_H(G)}$ *contains* $V_{\mathfrak{p}^2\uparrow G}(G)$.

**Proof.** We note from (3), that $V_{\mathfrak{p}^2\uparrow G}(G)$ contains $\tilde{R}_1^*$ (cf. (10)), hence

$$V_{\mathfrak{p}\uparrow G}(G) = \mathrm{SV}_{\mathfrak{p}^2\uparrow G}(G)\tilde{R}_1^*.$$

We have the commutative diagram

$$\begin{array}{ccccccccc}
 & & D & \xrightarrow{\;\sim\;} & \tilde{R}_1^* & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & \operatorname{Ker} \varphi' & \longrightarrow & V_H(G) & \xrightarrow{\varphi'} & H & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \operatorname{Ker} \varphi'_S & \longrightarrow & \mathrm{SV}_H(G) & \xrightarrow{\varphi'_S} & H & \longrightarrow & 1
\end{array}$$

Hence $D=\tilde{R}_1^*$ and Ker $\varphi'=$ Ker $\varphi'_S D$.

We note that $\tilde{R}_1^* \subset$ Ker $\varphi'$ (10). To see this, observe first that

$$\varphi(\tilde{R}_1^*)=\varphi(\tilde{R}_1^*\Gamma_m(\mathfrak{p}^2)/\Gamma_m(\mathfrak{p}^2))$$

has order 1 or $p$, in the latter case we have $\varphi(\tilde{R}_1^*)=H$. However, $\tilde{R}_1^*\Gamma_m(\mathfrak{p}^2)/\Gamma_m(\mathfrak{p}^2)$ is clearly centralized by the group $C$ which is represented diagonally in $\mathbb{Z}Ge_0/\Gamma_m(\mathfrak{p})$. So we must have that $\varphi(\tilde{R}_1^*)$ commutes with $\varphi(C)=C$. This gives $\varphi(\tilde{R}_1^*)=1$. Now it follows

$$V_{\mathfrak{y}^2\uparrow G}(G)=\mathrm{S}V_{\mathfrak{y}^2\uparrow G}(G)\tilde{R}_1^* \subset \mathrm{Ker}\ \varphi'$$

by Claim 8.

This also proves Proposition 2. □

We now are finally in the position to prove that our splitting $\varphi$ comes 'essentially' from an ideal.

From the proposition it follows that we obtain a $V(C)$-homomorphism – note that $V_H(G)/V_{\mathfrak{y}^2\uparrow G}(G)$ is abelian:

$$\bar{\varphi}: V_H(G)/V_{\mathfrak{y}^2\uparrow G}(G)\to H.$$

Combining this with the $V(C)$-isomorphism from Claim 1(i)

$$V_H(G)/V_{\mathfrak{y}^2\uparrow G}(G)\simeq \mathfrak{y}\uparrow^G/\mathfrak{y}^2\uparrow^G,$$

we conclude that the $V(C)$-action on $H$ induced from $\varphi$ must be the same as the $V(C)$-action on $H$ induced by one of the ideals $I_j$, $1\le j\le m$ (Claim 3), and so there must also exist a splitting induced by $I_j$ for some $1\le j\le m$, such that both $V(C)$-actions are the same. Applying this to our examples we conclude:

**Proposition 3.** *For the Frobenius groups $C_{73} \rtimes C_8$ and $C_{241} \rtimes C_{10}$ there can not exist a homomorphism $\varphi: V(G)\to G$ splitting the natural injection $G\to V(G)$.*

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